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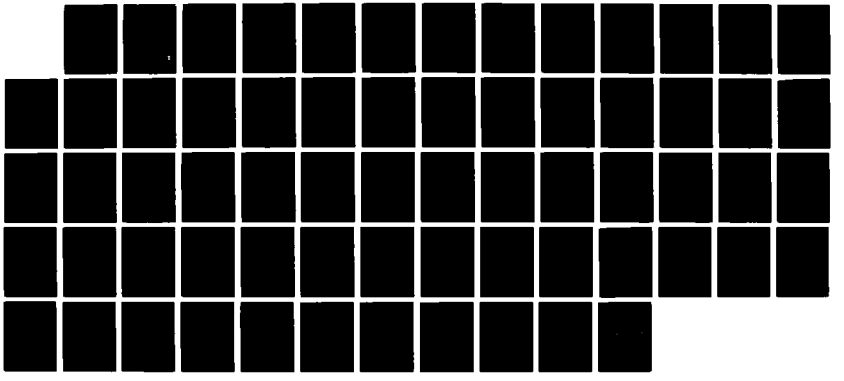

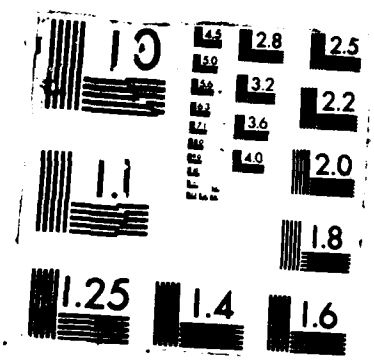
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TOWARD AN INTEGRATION OF ITEM-RESPONSE THEORY AND COGNITIVE ERROR DIAGNOSES

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Introduction

Finding the sources of misconceptions possessed by students is a difficult task because it is impossible to see what is happening in their brains. The only observable outcomes are the students' responses on the test items. Studying their spoken-aloud protocols is one method for discovering how students solve or think through problems. Several programs that are capable of diagnosing students' misconceptions have been developed in the past decade (Brown & Burton, 1978; Marshall, 1980; Vanlehn, 1983; Tatsuoka, Baillie, & Yamamoto, 1982; Sleeman, 1984; Ohlsson & Langley, 1985). The common ground of these cognitive diagnostic systems is that they infer unobservable cognitive processes from logical interrelationships among cognitive tasks, subtasks and goals involved in the domain of interest. It is important that we retrieve invisible things from a "black box" and interpret them into a useful form so that valuable information can be obtained for improving educational quality.

The effort of incorporating the findings in cognitive theories into construction of test items is not new. Bloom and his associates (1956) created a taxonomy of educational objectives for the cognitive domain and it has become very popular among educational practitioners in the past decades. However, some people have had considerable difficulty in classifying items according to Bloom's taxonomy. Statistical properties of the taxonomy (Madaus, Woods & Nuttall, 1973; Miller, Snowman, & O'Hara, 1979) were investigated and the

results suggested only two factors, fluid and crystallized intelligence (Cattell, 1971).

Recent advances in cognitive theory provide new insights into human thinking and learning processes. As a result, it had become apparent that the need for a new kind of test has arisen. The new tests have to be used as an integral part of instruction. In order to be of any use, the measures of the new assessment must promote learning and improvement of instruction. Reshaping of testing will lead to better measures of ability and achievement with greater instructional utility.

Linn (1985) pointed out that "there are a number of barriers that will need to be overcome if the envisioned improvements in testing are to be even partially realized." He describes three barriers: a lack of technical, methodological theories that are appropriate to handle dynamic aspects of modern learning theory; the economic barrier; and the ideological barrier. As for the latter barrier, the researchers and practitioners dealing with current theories of educational and psychological statistics perceive "noise" or "response errors" as an incidental factor, and are not accustomed to seeking reasons for what they are and why they happen. Aberrant responses are seen as errors, although they often coincide with the responses resulting from the perfect application of wrong rules (Birenbaum and Tatsuoka, 1982). Linn (1985) says that "it is important to understand the obstacles to change, whatever their nature, in order to overcome them."

In this study, a new methodology that is capable of diagnosing cognitive errors and analyzing different methods for solving problems

will be introduced, illustrated with fraction subtraction problems. The new approach called "Rule space" integrates Item Response Theory (IRT) and the algebraic theory of databases (Lee, 1983). The rule space model is general enough to apply to any domain of interest, where classifications and selections of students, diagnoses of misconceptions are done by Bayes' decision rules for minimum errors (Fukunagawa, 1972) or any other equivalent decision rules with respect to a set of systematic errors determined prior to analyses. The first section discusses important objectives which the construction of cognitive diagnostic tests must follow. Then the introduction of the rule space model starts with its brief concept, connection to a distribution theory, construction of a bug library and the rule space, and finally introduction of the operational classification scheme.

On the Construction of Items for Cognitive Diagnostic Testing

Selection of the items in cognitive diagnostic testing is important. Special care has to be taken in selection of the items. For example, one of the most popular misconceptions in the fraction subtraction problems committed by seventh or eighth graders (2.8% of the total number of students), is when it is necessary to increase the numerator of the first fraction of a subtraction problem, a student reduces the whole number part by one and adds 10 to the numerator of the fraction (Shaw, 1984). If the denominator of the first fraction happens to be 10, then the procedure produces the correct answer. Therefore, at least one item must have a denominator

not equal to 10 in the first fraction, while a second item may have the denominator of 10. Then, the response pattern (1 for the correct answer and 0 for wrong answers) yielded by the right rule is (1,1), but the rule adding 10 to the numerator produces the response pattern of (1,0) or (0,1). .

Traditionally, item construction originated from the evaluation of content validity, that is, "how well the content of the test item covers the class of situations or subject matter about which conclusions are to be drawn" (APA...). However, Angoff (1986), Messick (1980) and Cronbach (1985) came to the same conclusion as the view offered by Loevinger in 1957: "Content validity is essentially ad hoc and does not have scientific value." The recent view of content validity, therefore, replaces its essence by construct validity.

Construct validity is termed so as to "examine the psychological trait, or construct, presumed to be measured by the test and investigate relationships between the data from the test and the theory underlying the construct" (Messick, 1984). Futher, Angoff (1986) states "Construct validation is a process, not a procedure; and it requires many lines of evidence, not all of them quantitative." According to his view, a validation study may include logical task analyses such as that done by Klein et al. (1981) for predicting bugs or erroneous rules of operation and construction of test items by which all the predicted bugs will be covered. Protocol studies have also become an important component of the validation process.

Glaser (1985) suggested a new direction of educational measurement in achievement test. He writes "Test items can be comprised of two elements--information that needs to be known and information about the conditions under which use of this knowledge is appropriate." As for the first element, there are various stages of competence in students' knowledge, including cognitive skills. Also, it is important to assess what knowledge structure the students have. Greeno (1980) pointed out that the acquisition of declarative and procedural knowledge is usually an object of instruction but strategic knowledge that enables one to set goals and subgoals and to form plans for attaining goals is not explicitly taught. It is often left to individuals to acquire it by induction. Different item types often require the students to decide which solution path should be taken, what should be done first to reach the final answer. Many erroneous rules discovered in the studies of signed-number and fraction addition and subtraction problems indicated the students did not even recognize that different item types require different solution paths and different sequences of different subtasks. They did not have the slightest idea of why the strategic skill described by Greeno were necessary.

The new test design must be capable of reflecting and discriminating between the different knowledge structures possessed by individuals. Each different structure requires its unique strategic skills. For example, there are two contrasting methods to solve mixed number fraction subtraction problems. One is to separate whole number parts from fraction parts (Method B) and the other is to

convert mixed numbers to improper fraction first and then subtract two numbers (Method A). If a student has excellent computational skills in the arithmetic of whole numbers, then he/she does not have to learn how to borrow one from the whole number part. So, Method A always gives the correct answers for subtraction of mixed numbers. The student using Method A can get high scores without understanding numbers and the number system.

Modern cognitive theory concludes that one of the important stages in learning processes is "Theory change" (Glaser, 1985). When learning takes place, students test their hypotheses and then evaluate, modifying current theories on the basis of new information. New educational measurement must take the volatile theory changes by an individual into account and be able to capture the traces of performance changes in detail in order to increase educational utility of responses to the tests.

The goals to be attained in modern measurement theory are not easy. Apparently, the technical barrier, as Linn states, is high and the traditional theories of educational measurement and testing have only limited power, or are simply inapplicable to the new measures. IRT is not an exception in dealing with the new demands just described if we conceptualize IRT at the level of individual items.

The definition of construct validity, thus, has to be expanded to a reconceptualization of traditional test theory so as to explain the dynamic aspects of learning and to express knowledge structures in terms of relational databases. None of the traditional test theories can handle theory changes or assess knowledge structures.

As for a summary of the above discussions, Glaser (1985) categorizes the main objectives of assessing new achievement measures into four parts: The first is to diagnose the principles of performance, the second is to assess the theory changes, the third is to evaluate a structure or representation of problems, and the fourth is to assess the automaticity of performing skills. The automaticity is important to reduce attention-demanding tasks. Carrying out single component processes may be easy, but it may not be easy when several components have to be worked out together. The importance of this operation, orchestration of several component tasks, seems often to be neglected by textbooks and in classroom teaching.

The four dimensions of objectives listed above seem very descriptive and qualitative. Psychometrics, by nature, is concerned with quantitative theories of educational and psychological measurement. The area of standardized testing has a long history of contributions to American education. The development of IRT has led to many areas of improvement, such as item analysis, test design, test equating, and procedures for detecting item bias.

The basic concept underlying IRT is latent traits. "A theory of latent traits supposes that an individual's behavior level can be accounted for, to a substantial degree, by defining certain human characteristics called traits, quantitatively estimating the individual's standing on each of these traits, and then using the numerical values obtained to predict or explain performance in relevant situations" (Lord and Novick, 1968, p. 358). Messick (1984) projected his view of new achievement assessment as a combination of

trait theory and the decriptive theory used in differential psychology. However, there is no guarantee that traits exist in any physical or physiological sense. "It is sufficient that a person behaves as if those amounts substantially determined his behavior" (Lord and Novick, 1968, p. 358). Convincing interpretation trait variables have never been given in the past.

Returning to the definition of construct validity given by Angoff (1986), let us examine the psychological traits or constructs as viewed by traditional psychology. Catell describes ability as organized complexes of transferable concepts and skills. Guilford (1967) describes it as information processing skills. Sternberg (1977) treats abilities as constellations of information-processing components. Snow (1986) conceives of abilities as structures of assemble and control processes as well as performance processes.

On the other hand, psychometricians have developed probabilistic models to measure invisible constructs. Two general classes of the models have been proposed: continuum and state models. Two conflicting views are at the root of the models: For continuum models, trait acquisition is assumed to be continuous in nature, while state models take the position of all-or-none, discrete processes.

As for the continuous models, several pioneers have developed various types of item response theory models (Lord, 1953; Lord and Novick, 1968; Birnbaum, 1968; Rasch, 1960). Since then, several extentions or modifications of the original IRT models have been developed (Samejima, 1969; Bock, 1972; Fischer, 1973; Embretsen,

1984; Reckase, 1985). The IRT models express a trait by the variable θ and express item characteristics by continuous probability functions. That is, item scores x_j(j=1,2...n) are related to a trait θ by a function that gives the probability of each possible score on an item for a randomly selected examinee of given ability. These functions are response curves of item characteristic curves. Although the models fit the datasets of various standardized tests, the clear meaning of θ remains undetermined through data analysis. Spada and McGaw (1985) investigated cases for which the one-parameter logistic model and Fischer's linear logistic model (LLTM) are inapplicable. They concluded "the simple Rasch model is applicable only if global learning of item-specific learning occurs, with constant gains for all persons. Person specific learning falsifies the model. The same is true for application of LLTM that decompose item-specific learning into changes in the difficulties of elementary cognitive operations" (p. 189).

A discrete learning model pioneered by Lazarsfeld (1956) and Lazarsfeld & Henry (1968) also has branched out to many modified, extended models (Macready, 1982; Alvord and Macready, 1985; Muthén, 1985; Paulson, 1985). Paulson, in particular, extended the latent structure model to apply to detection of rules of operation in signed-number addition problems. Each rule was treated as a discrete state in his model. Some basic assumptions in the state models are too restrictive and unrealistic to incorporate into the modern learning theory. They assume a <u>priori</u> how many latent classes or states the model has. Then, every subject must belong to exactly one

of the finite sets of classes which are mutually exclusive and together exhaustive. The explanation of theory changes by state models is very difficult. The restriction in the number of states to be included in modeling before the parameters are estimated critically limits the flexibility of this approach. Although recent advances in methods of estimating parameters are significant, it is still an expensive task with respect to computer time and it requires a dataset of astronomical size in order to obtain accurate estimates of many parameters.

Tatsuoka and her associates (Tatsuoka, 1983, 1985, 1986; Tatsuoka & Tatsuoka, 1983, in press) have developed a new probabilistic model which has taken advantage of both the continuous and discrete models. They named it "Rule space."

Rule Space Model

1. Philosophy behind the rule space model

Whatever the invisible traits or constructs stand for, the statistical meaning of the estimated latent variable is equivalent to the proficiency levels of the performances on the test items because the total score or weighted total score is a sufficient statistic for estimating the true θ by the maximum likelihood method in the one- and two-parameter logistic models (two of the basic IRT models).

Suppose item j is scored 1 or 0; then x_j is a random variable related to θ by a probability function as follows:

$$(1) \quad P_j(\theta) = \text{Prob}(x_j = 1|\theta) = 1 - Q_j(\theta) = 1 - \text{Prob}(x_j = 0|\theta).$$

The one-parameter logistic function defines a basic type of model called the Rasch Model. Equation (2) gives the two-parameter logistic function, where a_j is item discrimination power, b_j is item difficulty

$$(2) \quad P_j(\theta_i) = 1/\left[1 + \exp(-1.7a_j(\theta_i - b_j))\right].$$

If a_j is set equal to 1, then it becomes the Rasch model.

Tatsuoka and Linn (1983) discussed the relationship between item response function $P_j(\theta_i)$ and person response function $P_i(b_j)$. The person response function (or curve) is defined by the same equation (2), but $P_j(\theta_i)$ is a function of a continuous variable θ for a fixed b_j. $P_i(b_j)$ is a function of variable b (assuming there are infinitely many items) for a fixed level of θ_i. Especially, the one-parameter logistic function is a symmetric function with respect to θ and b.

The person response function is the probability function of person i with $\theta_i = \theta$ getting the correct answer for an item with difficulty b_j. Since Mosier explored person response curves in 1941, several researchers have found them very useful for explaining the relation between ability θ and item difficulty b (Trabin & Weiss, 1979; Carroll, 1985). By using the one-parameter logistic model, Carroll (1985) explored the relation between both the curves with several ability tests in order to obtain an answer to the question, "what is an 'ability'?" (p. 1). His assumption is that "The existence of an ability can be demonstrated when it can be shown that for any individual there is a systematic, monotonic, and close

relation between the individual's probability of correct or satisfactory performance and the difficulties of a series of tasks, and when there are variations over individuals in the parameters of this relation." (p. 22). One of his conclusions is that the ability is defined in terms of the attribute(s) of the tasks that cause differences in task difficulty. Carroll's conclusion is that the ability is defined in terms of the attribute(s) of the tasks that cause differences in task difficulty. Carroll's conclusion is applicable to the situation such that the space of the difficulties resulting from various combinations of attributes involved in the tasks is unidimensional. Then his conclusion is mathematically sound because of the symmetric relation of the Rasch model.

As long as we interpret θ as the latent ability or construct which influences the performances on the tests, we may face the philosophical dilemmas of IRT models such as the dimensionality of θ or b, or the impossibility of explaining gain scores or Glaser's Theory changes. Since a composite of several factors (or abilities) influences the performance of an item, and each item in the test is likely to require a different composite or possibly a different set of abilities, the psychological meaning of θ is very complicated and difficult to interpret (Stout, in press).

IRT models are formulated by utilizing a response pattern or binary vector with n elements. There are two distinct independent pieces of information in a response pattern. One is quantitative, telling us how many items are correctly answered (total score), and the other is qualitative, regarding which items are correctly

answered and which are missed. Since IRT assumes the local independence (Lord & Novick, 1968), the likelihood function is expressed by taking the two pieces of information into account. The item and person parameters can be estimated from the likelihood function by applying the maximum likelihood procedure.

However, Tatsuoka (1987) investigated what really determines the item response curves, a task somewhat similar to Caroll's quest regarding "what ability is." Starting from a painstaking logical task analysis, she constructed sets of items such that each item involves a unique combination of cognitive subtasks. The study showed that underlying cognitive processes for solving an item determine the slope and location parameters of an IRT curve. If this is true for any domain, then it is important to expand our view from a localized, narrow interest in IRT curves to a broader global concept. Refinement and improvement in psychometric techniques, after IRT models were introduced, have been limited for most works to dealing with individual items separately.

A new model has to be able to measure the objectives of modern learning theory. In order to achieve this demanding goal, it is helpful to see item response curves as a whole, as a set, and to investigate algebraic and topological properties of this function space. Many IRT models have been proposed recently. These probability functions provide finer, or more accurate measure of θ than simple basic IRT models. They too can be used for formulating the rule space model, so it is not necessary to be restricted only to one- and two-parameter logistic models.

Expanding the perspectives of test theories to functional analysis leads to some important and useful conclusions. For example, Tatsuoka (1975) reformulated classical test theory in a vector, Hilbert space and proved the existence of the true score.

The theory of functional analysis treats a function as a "point" and utilizes the methods of algebra and topology in a set of functions. Ramsay (1982) outlined the concept of functional analysis as an extension of classical statistical techniques and explained least squares, principal components and canonical correlation analyses in the context of functional analysis. He stated that "the data must be viewed as an element in a space of possible functions taking a domain space into a range space" (p. 352). He concludes with the remark that "functional analysis already has revolutionized numerical analysis so that any issue of a major journal [in that discipline] now has a number of papers using this technology. I claim that this is about to happen to statistics" (p. 394). The rule space model is an application of functional analysis using a projection operator.

The third objective listed by Glaser was to assess the structure of knowledge possessed by an individual. This dimension requires a leap to a new world for the field of educational measurement, as it is apparent that the algebraic theory of relational databases is needed to achieve this objective. If a set of items is carefully constructed, then various relationships among the items should reflect the relationships among cognitive subtasks underlying each item. By examining bugs and sources of misconceptions, one can see

which subtasks caused scores of ones or zeros on the items. For example, Tatsuoka (1984c) represented 27 erroneous rules of operation found in signed-number addition problems as ordered pairs of (1) the number of cognitive steps taken correctly, and (2) the value of the Norm Conformity Index, one of several indices that measure appropriateness of response patterns (Tatsuoka & Tatsuoka, 1982). The latter assesses the degree of conformity of the sequence of cognitive steps followed by each bug to the expert's procedural steps. That is, the second value measures how early or how late an erroneous rule causes departure from the correct steps in the procedural network. The result of the study indicated that an early derailment obviously has more serious consequences than one at a later stage. Cluster analysis separated bugs caused by early and frequent derailments from those due to later and less frequent derailments. She named the former, "seriously ill-composed rules." Later studies (Birenbaum & Tatsuoka, 1986a; Shaw, 1986) indicated that wrong rules that are not seriously ill-composed can be remediated by giving correct answers, or even by the feedback of OK or NO to each reponse to the item.

Norm Conformity index characterizes the quality of information of a response pattern and expresses its characteristics quantitatively. The rule space uses a similar index (called IRT-based caution index) defined in the context of the IRT curves (Tatsuoka, 1984b). However, in order to explain what the rule space model is, we have to clarify the terms "bug" and "erroneous rules of operation," which have been used without specific definitions so far.

2. What are Rules?

A systematic error over the test items can be the combination of one or more bugs, or a part of erroneous rules. If a student applies his/her erroneous rule with perfect consistency to the items in the test, then his/her responses to the test will be perfectly matched with the responses generated by a computer program. We call systematic errors erroneous rules.

A correct rule will, by definition, produce the right answer to all the items, but sometimes wrong rules may produce the right answer to some subset of the test items. Moreover, some rules are combinations of the right rule and wrong rules, and others are combinations of two or more different wrong rules. We consider them as new rules as long as they are consistently applied to all the items. If we construct the test items carefully so that the important, predicted common errors can be expressed by unique item response patterns of ones and zeros, or component response patterns specified in a task analysis, then the rules can be distinguished by response patterns. In other words, we can assume that rules are expressed by binary response vectors.

The rule space model was developed to solve a specific classification problem in which the entities to be classified are the rules in some well-defined domain such as arithmetic, algebra and science. As was mentioned earlier, two kinds of information are included in a response pattern. They are the quantitative information of the total number of ones and the qualitative information on which items had ones. The former is represented by θ,

the latter, by ζ, an index expressing atypicality (or non-appropriateness) of the response patterns of a given group (Tatsuoka, 1984b). Any rule can be expressed by an ordered pair of θ and ζ. We assume that, at least at the very beginning, the rules are determined a priori through a logical task analysis. A later section explains this in detail.

3. Bug Distribution

The distribution of observations plays an important role in statistical theories. When we deal with students, random errors or slips due to careless errors or uncertainty always affect the outcomes of performances on a test. Even if a student possesses some systematic error, it is very rare to have the response pattern perfectly matched with the patterns theoretically generated by its algorithm (VanLehn, 1983; Tatsuoka, 1984a). Some systematic errors may have a tendency to produce more slips, while other rules have a small number of slips. Also, some items may be prone to produce more slips than other items. It is very important that we be able to predict the probability of having slips on each item for each systematic error (or rule). Knowing the theoretical distribution of observed slips of a rule enable us to see and predict statistical properties of observed responses yielded by the rule.

Tatsuoka and Tatsuoka (1987) derived the theoretical distribution of observed slips and named it "Bug Distribution." First, the probability of having a "slip" on item j (j=1,2,...,n) is denoted by p_j for item j, (j=1,2,...,n) and it is assumed that slips occur independently across items. The bug distribution of a rule R

follows a compound binomial distribution with different slip probabilities for the items.

$$(3)\quad \text{Prob (having up to s slips}|R) = \left\{ \sum_{m=0}^{s} \left\{ \sum_{\Sigma x_j = m} \prod_{j=1}^{n} p_j^{x_j} (1-p_j)^{1-x_j} \right\} \right\}.$$

The expectation and variance of the bug distribution of Rule R whose corresponding binary vector is $\underset{\sim}{x}_R$--in which we assume, without loss of generality that the first r elements are ones and the remainder are zeros (i.e. $x_{r_j}=1$ for $j=1,\ldots,r$ and $x_{r_j}=0$ for $j=r+1,\ldots,n$)-- will be given by

$$(4)\quad \mu_R = \sum_{j=1}^{r} p_j + \sum_{j=r+1}^{n} q_j$$

$$(5)\quad \sigma_R^2 = \sum_{j=1}^{n} p_j q_j$$

where $q_j=1-p_j$.

The bug distribution of rule R corresponds to the conditional probability that a subject in the state of possessing rule R will respond correctly to item j.

4. **A Mapping Function of Response Patterns $\underset{\sim}{x}$.**

The rule space model begins by mapping all possible binary response patterns, $\underset{\sim}{x}$'s into a set of ordered pairs $\{(\theta, f(\underset{\sim}{x}))\}$. The mapping function $f(\underset{\sim}{x})$ is an inner product of two residual vectors, $\underset{\sim}{P}(\theta) - \underset{\sim}{x}$ and $\underset{\sim}{P}(\theta) - \underset{\sim}{T}(\theta)$ where $P_j(\theta)$ $j=1,\ldots n$ are the logistic

functions and $\underset{\sim}{P}(\theta) = [P_1(\theta),\ldots P_n(\theta)]$ and $\underset{\sim}{T}(\theta) = (T(\theta),\ldots T(\theta))$, $T(\theta)$ being the average of $P_j(\theta)$, j=1...,n. Since $f(\underset{\sim}{x})$ is a linear function, the bug distribution of Rule R mapped into the rule space will have the centroid of Equation (6) and the variance and covariance matrix of Equation (7) (Tatsuoka, 1985).

$$(6)\quad f(x_R) = -\sum_{j=1}^{r} Q_j(\theta_R)\left[P_j(\theta_R) - T(\theta_R)\right] + \sum_{j=r+1}^{n} P_j(\theta_R)\left[P_j(\theta_R) - T(\theta_R)\right]$$

$$(7)\quad \Sigma_R = \begin{bmatrix} 1/I(\theta_R) & 0 \\ 0 & \sum_{j=1}^{n} P_j(\theta_R)Q_j(\theta_R)(P(\theta_R) - T(\theta_R) \end{bmatrix}$$

where θ_R is the θ-value for Rule R, $Q_j(\theta_R)$ is $1 - P_j(\theta_R)$ and $I(\theta_R)$ is the information function of the test at θ_R. The mapped bug distribution of R has the slip probabilities $S_j(\theta_R)$ given by Equation (8). See Tatsuoka and Tatsuoka (1987).

$$(8)\quad S_j(\theta_R) = (1 - x_{R_j})P_j(\theta_R) + x_{R_j}Q_j(\theta_R).$$

By standardizing $f(\underset{\sim}{x})$, IRT based caution index ζ (Tatsuoka, 1985) will be

$$(9)\quad \zeta = \frac{f(\underset{\sim}{x})}{\sum_{j=1}^{n} P_j(\theta_R)Q_j(\theta_R)\left[P_j(\theta_R) - T(\theta_R)\right]}$$

The mechanism of how the index ζ distinguishes atypical response patterns from typical ones is described in M. Tatsuoka and K. Tatsuoka (1986), so more detailed explanation than that given in

section 7.2 will not be given in this paper. But the statistical properties of the bug distributions will be discussed later.

The next section introduces an elementary relational structure of the items that will be used for preparing a list of erroneous rules in the bug library.

5. Differential Ordering of Items by Underlying Cognitive Processes.

If a content domain for constructing a cognitive diagnostic test is determined, then logical task analysis can nominate cognitive attributes. The attributes may refer to production rules, procedural operations, item types or, more generally, any cognitive subtasks. A set of n items can be characterized by K nominated attributes and expressed by K-element vectors. Let us call this matrix an attribute x item matrix, or Q-matrix. (Embretsen, 1984). It is hoped that the initial task analyses can be carried out by several experts or master teachers. If two experts use different methods to solve a given set of problems, then they may get entirely different attribute x item matrices.

In this study, all the nodes of a directed process network will be called "attributes" and denoted by $A_1, A_2, \ldots A_K$. Items will be characterized by placing one in the (j,k) cell of the Q-matrix when j involves attribute A_k, and a zero when item j does not involve attribute A_k.

$$q_{kj} \begin{cases} 1 \text{ if item j involves attribute } A_k, \\ 0 \text{ otherwise} \end{cases}$$

For example, Figure 1b, Method B is an attribute x item matrix for fraction subtraction problems solved by Method B (separate whole number and fraction parts).

Figure 1

The purpose of introducing an attribute x item matrix is, first, to make it easier to construct a set of items relevant to diagnosing students' misconceptions resulting from a lack of knowledge or misunderstanding of an attribute or combination of several attributes. The second aim is to extract a set of binary patterns of n items from the matrix, each pattern being produced by a systematic application of an erroneous rule resulting from a misconception or incomplete knowledge of a targeted attribute, or a combination of several attributes.

Starting from the attribute x item matrix, many researchers in a variety of disciplines such as biology, differential psychology, engineering, and sociology have investigated clustering techniques. Although the author has applied some techniques to the attribute x item matrix, the results were disappointing. By using a dataset, results of the analysis may be more objective, but interpretability of analysis results may be washed away to a great extent. The lack of interpretability of factors, obtained by factor analysis of the estimated trait variables is well known. It is a problem of trade-off between establishing objectivity and interpretability. At this stage of making a diagnostic test, we will take the value of interpretability of data into account. However, it is not our

intention to neglect objectivity. This issue will be discussed again in the summary and discussion.

Each item, represented by a column vector, Q_j of the attribute x item matrix is now characterized by a specific combination of attributes. Similarly, each attribute, a row vector $\underset{\sim}{Q}_k$ contains the information as to which items involve attribute k. Let R be a relation on a set of the column vector $\underset{\sim}{x}_j$, j=1,...,n, where n is the number of items.

Definition of R will be given as follows;

(10) $\underset{\sim}{x}_i \leq \underset{\sim}{x}_j$ if $x_{ik} \leq x_{jk}$ for k=1,...,K.

Or equivalently, if item j includes the attributes involved in item i then item j needs more task than item i. This relation satisfies the reflexive and transitive laws:

1) $\underset{\sim}{x}_i \leq \underset{\sim}{x}_i$, reflexive law

2) if $\underset{\sim}{x}_i \leq \underset{\sim}{x}_j$ and $\underset{\sim}{x}_j \leq \underset{\sim}{x}_l$ then $\underset{\sim}{x}_i \leq \underset{\sim}{x}_l$, transitive law.

With this relation, a set $\{\underset{\sim}{x}_1,...,\underset{\sim}{x}_n\}$, has a partial ordering. If the symmetric law is satisfied by $\underset{\sim}{x}_i$ and $\underset{\sim}{x}_j$ then, $\underset{\sim}{x}_i$ and $\underset{\sim}{x}_j$ are equivalent, and written as $\underset{\sim}{x}_i \sim \underset{\sim}{x}_j$. Let $\{\underset{\sim}{x}_j\}$ be a set of items expressed by vectors of K attribute elements where $\underset{\sim}{x}_j$, j=1,...n. From the attribute x item matrix a set of totally ordered items is extracted. When relation R exists for any two elements in a set S, then S is said to be totally ordered.

In Figure 1b, Method B, Items 6, 8, 12 and 10 are totally ordered but 6, 2, and 12 are not. Denote a set of totally ordered

items by S_i. Then a list of the totally ordered sets of the items extracted from Figure 1, Method B is listed in Table 1.

Table 1

In Figure 2b, Method B tree, a tree of the items is constructed from the list of totally ordered sets in Table 1. If two item are connected by a directed arc, then the items are totally ordered.

Figure 2

The number(s) in a box is (are) the item(s) involving attributes listed next to the box. An advantage to using an item tree is that a structural interrelationship among the items can be expressed schematically. With the process network, it is difficult to see all the different solution paths taken by 20 items in a single graph and also it is difficult to pinpoint why and where a student's response pattern deviates from the perfect responses. For example, Tatsuoka (1984) describes Rule 8 as "The student subtracts the smaller from the larger in corresponding parts when the two numbers are different." With rule 8, items 12 and 14 (11/8 - 1/8, 3 4/5 - 3 2/5) have correct answers while 2 and 17 (3/4 - 3/8, 7 3/5 - 4/5) don't. By marking each box in the item tree of Method B with * for the correct answers and x for the wrong ones, the sources of misconceptions producing rule 8 (refered to as G14) are represented clearly as can be seen in the following Figure 3--they are borrowing, getting a common denominator, and converting a whole number to a fraction.

Figure 3

6. Preparation of the "bug information bank" or "bug library."

Unlike most psychological models, the rule space model has been developed by emphasizing the importance of interpretability of statistics estimated from the data. Fischer and his associates (Embretson, 1985; Fischer, 1973) expressed item difficulties in the Rasch model by a linear combination of component subtasks, and also unobservable frequencies with which each component influences the solution of each item. Since the models contain several parameters in the logistic functions, the estimation of the item parameters has become a major task in the past ten years. (Fischer, 1978; Fischer and Formann, 1972). Scheiblechner (1972) estimated item difficulties from the Rasch model first and then regressed the estimated item difficulties onto the hypothetical frequencies contained in the task matrix Q. Estimated β-weights approximate fairly well the estimates of component subtasks obtained by a conditional maximum likelihood procedure. However, Spada and McGaw (1985) state, "Despite the value of the LLTM's analysis of task performance in terms of performance on elementary operations, there are some difficulties in interpreting the parameters of the model. The decomposition of the item difficulties is, of course, quite precisely defined but its psychological interpretation is not equally clear," (p. 180). The difficulties in interpreting the estimates of the psychological models are very common in psychometrics. It seems impossible to maintain the interpretability of estimated parameters in terms of underlying cognitive tasks in the current psychological modeling

approaches. New approaches must be flexible enough so that individual differences unaccounted for during the process of formulating a model will not only alter the interpretability of the estimates but also be able to determine the existence of subjects who don't fit the model. Since the item trees are constructed from the inclusion relationships among attributes, the interpretability is clearly retained.

6-1. Use of Multiple Regression

As did Scheiblechner (1972), a multiple regression analysis of the attributes onto the item difficulties of 40 items showed that the four attributes, converting a whole number to a fraction or mixed number, getting the common denominator, borrowing of whole-number subtraction and borrowing one from the whole number part to make the numerator larger, have significant β-weights to predict the item difficulties (Chevalaz, 1983). Also, the number of attributes involved in each item correlates with the item difficulty, at the value of .57. Therefore, it is true that the greater the number of attributes involved in the items, the more difficult the items are.

Suppose that a student cannot get the lowest common multiple of two denominators, and he/she uses Method A, but that the student can do the remaining attributes. Then, the response pattern of the performance on the 40 items will be 0 for the items involving the attribute cd in the Method-A item tree, 1 for those not involving the attribute cd. Because getting the common denominator has a substantial β-weight, this error can be a good indicator of determining a list of rules in a bug library.

6-2. Use of the Item Tree

There are 2^7 possible response patterns obtainable from the attribute x item matrix in each method. However, the item trees in Figure 2 enable us to select a smaller number of rules and bugs that are substantially important in designing and evaluating lessons.

The numbers shown near the directed arcs of Figure 2 are conditional probabilities, Prob $(X_i = 1 \mid X_{i-1} = X_{i-2} = \ldots X_1 = 1)$ where i-1, i-2, ...,1 are antecedent items of i, and X_i is the score of item i.

Since $\underset{\sim}{x}_i \geq \underset{\sim}{x}_{i-1}$ (i.e., item i includes all the attributes involving item i-1) a drastic decrease in the value of the conditional probability implies that a newly added attribute (or attributes) causes the change. For instance, for Method B, the new attribute added to item 17 is indeed a difficult subtask, borrowing. If a student can't do the new attribute, borrowing, and can do the other attributes perfectly well, then subsequent items not including the borrowing attribute can be answered correctly. Thus, a binary response pattern corresponding to the student's performance will be zeros for the items that involve borrowing and ones for non-borrowing items. This conjecture with respect to borrowing will be confirmed by examining the arc between items 12 and 10.

Next, the conditional probability value between items 9 and 7 is .60, which is low enough to merit attention. The new attribute in the second box is "w to f or m" -- converting whole numbers to fractions or mixed numbers. Therefore, the second response pattern resulting from this case is zeros for the items with whole numbers

such as 3 - 1/2, and ones for the items not including whole numbers in their first position. By proceeding in this manner a set of response patterns that are logically interpretable will be obtained. Thus, representing the structural relationships among the items with respect to their underlying cognitive attributes facilitates error analysis. We have developed a computer program to make an item tree from an attribute x item matrix and extract a set of totally ordered items. Then, applying the method just mentioned, a list of 39 rules is prepared. They are coded as G1 through G39. For instance, G2 is the binary pattern of ones for items 6, 8, 26, and 28 and G13 is binary pattern of ones for easy non-borrowing items. The interpretation of G2 is that a student can subtract two numbers if no attribute in Figure 1 is involved. Then, their centroids and variance-covariance matrices are stored in the bug library for later use.

7. Stochastic Behavior of the Rules: Inferences from the Bug Distribution

7-1. Which rules are More Consistently Applied?

The bug distribution of Rule R defined on the neighboring response patterns of R was introduced in the previous section. It was expressed by a compound binomial distribution with the slip probabilities $S_j(\theta_R)$, j=1,...,n. Let $S(\theta_R)$ be the mean of $S_j(\theta_R)$, j=1,...n. Since any rule can be used as R, θ, instead of θ_R will be used. Walsh (1954) expanded the compound binomial in powers of $S_j(\theta) - S(\theta)$. Suppose X is a random variable of slip and s is the number of slips, then the probability of having s slips is given by (11),

(11) $\text{Prob}(X=s) = P_n(s) + 1/2\ nV_2C_2(s) + 1/3\ nV_3C_3(s) + (1/4\ nV_4 - 1/8\ n^2V_2^2)$

$C_4(s) + (1/5\ nV_2 - 1/6\ n^2V_2V_3)C_5(x) + \ldots,\ s = 0,1,2,\ldots,n,$

where $P_n(s) = \binom{n}{s}S(\theta)^s\ (1-s(\theta))^{n-s}$ for $s = 0,1,2,\ldots n,$

$$C_r(s) = \sum_{v=0}^{r} (-1)^{v+1}\left[\begin{matrix} r \\ v \end{matrix}\right]P_{n-r}(s-v),$$

$$V_r = 1/n\left[\sum_{g=1}^{r}\left[S_j(\theta) - S(\theta)\right]\right]^r,\ r = 2,3\ldots,n.$$

Let us use G2, and G13 for illustrating the relationship between the rule the probability of having s-slips away from the rules.

Insert Table 2 about here

As can be seen in Table 2, the probability of having a slip at item 6 is .610 for Rule G2 and .240 for G13. Since the scores for item 6 of both G2 and G13 are 1, the probability of having a slip, 1 to 0, at item 6 is higher for G2 than for G13. That is,
$\text{Prob}\ (X_6 \neq X_{G2_6} \mid G2) = .610$ and $\text{Prob}\ (X_6 \neq X_{G13_6} \mid G13) = .240.$
Item 6 produces more slips for G2 than for G13. Table 3 shows the

Insert Table 2 about here

theoretical frequency distributions of Rules G13 and G2. The probability of having 5 slips for G2 is .225 and for G13 is .150. The frequency distribution of G13 reaches the mode at s = 6 while that of G2 reaches the peak at s = 5. A close examination of the bug distributions indicates that as θ comes closer to the sample mean, the rule associated with such θ produces more slips. As θ becomes larger or smaller, such rules produce fewer slips. This

gives rise to the conjecture derived from theoretical bug distributions with the results of error analysis performed on fraction addition problems as did Tatsuoka (1983) with signed-number subtraction problems.

7-2. Which Rules are Atypical?

The IRT models assume the local independence of reponses to the items. Therefore, the likelihood of each rule can be computed by Equation (12). For each rule R_i,

$$(12)\quad L_{G_i} = \prod_{j=1}^{n} P_j(\theta)^{R_{ij}}(1 - P_j(\theta)^{1-R_{ij}}.$$

It is known that the likelihood correlates very highly with ζ (Harnisch & Tatsuoka, 1983; Birenbaum, 1985, 1986). The numerator of ζ is the function, $f(\underset{\sim}{x})$, linear function of $\underset{\sim}{x}$, $K(\theta) - \underset{\sim}{x}\left[P(\theta) - T(\theta)\right]$ for a fixed θ, where $K(\theta) = P(\theta)\left[P(\theta) - T(\theta)\right]$ is a constant for a given θ.

$$(13)\quad f(\underset{\sim}{x}) = \underset{\sim}{P}(\theta)\left[\underset{\sim}{P}(\theta) - \underset{\sim}{T}(\theta)\right] - x\left[\underset{\sim}{P}(\theta) - \underset{\sim}{T}(\theta)\right]$$

Since IRT curves represent underlying cognitive processes as mentioned earlier, the deviation of $P(\theta)$ (denoted by $P(\theta) - T(\theta)$) also reflects those cognitive processes. But the shapes of function $p_j(\theta)$ will be very different from those of the original logistic functions. Figure 4 shows three curves where the difficulty of item 27 is smaller than the average function $T(\theta)$, and that of item 28 is

greater than $T(\theta)$. Let us take $T(\theta)$ as the horizontal axis and the value of $P_j(\theta)$ as the vertical axis and draw the deviations of item response curves. Then graphs of deviations $p_j(\theta)$ will be as in Figure

Insert Figures 4 & 5 about here

5. Easier items are located in the upper half of the space while more difficult items are in the lower half. If a student with ability θ takes the score of 1 for easier items and 0 for harder items then the value $f(\underset{\sim}{x})$ will be smaller. If he/she scores 0 for easier items and 1 for harder items, then the values of $f(\underset{\sim}{x})$ tend to become larger. The same relation will hold for Equation (12), the likelihood function.

For example, if a student has a wrong rule for the borrowing operation in fraction subtraction, then his/her response pattern consists of ones for the items which do not require borrowing and zeros for those requiring borrowing. Figures 6 and 7 show two sets of strikingly different curves of $p_j(\theta)$. The first set of items, in Figure 6, contains items 4, 10, 11, 13, 17, 18, 19, and 20 which require borrowing before subtraction of the numerators is carried out. The second set of items, Figure 7, includes non-borrowing items 1, 2, 3, 5, 6, 8, 9, 12, 14, and 16 in which items 7 and 15 are excluded. Items 7 and 15 need conversion of a whole number to improper fraction or mixed number. All the items in Figure 6 (except item 4) have the functions $p_j(\theta)$ below the x-axis which is the average function $T(\theta)$. The functions in Figure 7 all have curves above the horizontal axis.

Insert Figures 6 & 7 about here

The response pattern associated with the borrowing error has the following binary vector of:

G19 = (11101101100111010000111011011001110100000)

and the maximum likelihood estimate of the latent variable θ is .09. The value of $T(\theta)$ for G19 is obtained by substituting θ = .09 into the item response functions and yields T(.09) = .48. Next, let us examine the values of the functions $P_j(\theta) - T(\theta)$ at T(.09) = .48. As can be seen in Figures 6 and 7, the functions $p_j(\theta)$ at Tθ = .48. have clearly different values for borrowing versus non-borrowing items. Since the value of Equation (13) at a given θ depends on the item score of 1 or 0, the ζ value of G19 will be a negative number (because the value of $K(\theta)$ is nearly zero). By dividing the value of f(G19) by the standard deviation at θ the value of Equation (9) is obtained. Thus, G19 corresponds to (.09, -2.26) in the rule space.

The rule space model starts by mapping all possible binary response patterns into a set of ordered pairs $\{(\theta,\zeta)\}$ and representing it by a Cartesian product space. A list of bug-response patterns {Gr, r=1,...39}produced from the item-tree program (Baillie & K. Tatsuoka, 1985) will be expressed by a set of ordered pairs (θ_r, ζ_r) which are centroids of r binary patterns. Table 4 lists the 39 points.

Insert Table 4 about here

Figure 8 shows the 39 centroids of 39 bugs in the rule space; their values are given in Table 4. In Figure 8, the cluster

Insert Figure 8 about here

represented by circled + signs are the bugs derived from the item tree by Method A, and the squares are those by Method B. It is interesting to see that the two sets of bugs derived from the two entirely different structures of the item trees partition the rule space, and yet spread evenly over the θ-axis. Since the points below the θ-axis conform better to the order of item difficulties, they are more typical performances on the test items. If a rule is very unusual, then the location of the rule will be seen in an upper part of the space. So, the location within the space will tell whether or not the rule is atypical with high scores or low scores. The same is true for typical rules with high or low θ's. Figure 9 contains a selected set of ellipses whose major and minor axes are $1/I(\theta)$ and 1, respectively (Tatsuoka, 1985).

Insert Figure 9 about here

Operational Classification Scheme

If all rules prepared in the bug library are mapped into a set of ordered pairs, $\{(\theta_R, \zeta_R)\}$ along with their neighboring response patterns with several slips away from each rule, then the topography would be like Figure 9.

The population of points would exhibit modal densities at the rule points, and each rule forms the center of an enveloping ellipse with the density of points getting rarer as we depart farther from

the center in any direction. Furthermore, the major and minor axes of these ellipses are parallel to the vertical and horizontal reference axes, respectively. The set of ellipses with Mahalanobis distance as the metric, gives a complete characterization of the rule space. If the ellipses represent misconceptions possessed by a majority of students then any response-pattern point can be classified as most likely being a random slip from some rule. For a students' response-pattern point we search the two nearest ellipses by computing the Mahalanobis distances of the student's point to the centroids of the ellipses. Then, Bayes' decision rule for minimum error is applied to classify the point and to determine error probabilities.

However, computation of error probabilities (the probability of misclassifications) is not an easy task. If the two nearest ellipses from the student's point have equal variance-covariance matrices and if the mapped distributions of the response patterns around the rules R_1 and R_2 into the rule space follow the multivariate normal distributions, then the logarithm of the likelihood-ratio function becomes linear. Therefore, computation of the error probabilities is reduced to the integration of the posterior conditional density functions (Tatsuoka & Tatsuoka, in press). More detail of the general procedure will be found in Fukunaga (1971). The assumption of equal variance-covariance matrices of R_1 and R_2 must, needless to say, be thoroughly examined.

1. The Results of Classification

The 535 students' responses on the 40-item fraction subtraction problems are mapped into the rule space, and almost 90% of the

students are classified into 39 sources of misconceptions. Table 5 summarizes the number of students classified into the ellipses derived from the item trees of Method A and B in Figure 2.

Insert Table 5 about here

The distribution of Method A users and Method B users over the θ-axis, is fairly even and not much different in terms of the level of θ's. In an earlier study by Tatsuoka (1984), 275 out of 535 students, most of them seventh graders in a local junior high school, had never been taught Method B. The remaining 260 students were from a different school and most of them were in the eighth grade. Moreover, quite a few students used both methods. If they don't need borrowing, then they use Method B, otherwise they switch their method from B to A (Shaw, 1984, p. 43). Method A does not require the borrowing skill. These students are left undiagnosed in this study, because the item trees utilized in producing a list of bugs are of either Method A or B and the combination tree of Methods A and B is not constructed. One of the advantages of using the rule space model is that uncertain errors on the test can be left classified without forcing them into one of the ellipses. They can subsequently be further investigated for their underlying cognitive processes. Therefore, the model can be exploratory and sensitive to any changes which affect students' performances.

Regarding the psychological meaning of θ, the content of Table 5 is investigated in detail. The histograms of both groups over the θ-axis are drawn and shown in Figure 10.

Insert Figure 10 about here

The θ values in the two groups are not significantly different but the ζ values are different. Histograms 1 and 2 for θ overlap and cannot be distinguished from each other. However, the Method-B users have a deeper understanding of the number system and many of them later advance to an algebra class. If θ is really an ability influencing the scores of the 40-item fraction subtraction test, it is only natural to assume that the Method-B users should have higher ability levels than those of the Method-A users. As Resnick (1982) states, Method A requires a better short term memory with accurate computational skills, while Method B requires sophisticated manipulation of the numbers.

Tatsuoka, Linn and Yamamoto (1986) analyzed the dataset more carefully and found that Method B users had higher mean scores for most subtasks except for borrowing. They clearly demonstrated that borrowing caused differential item performances. This study indicates that the rule space model is useful for studying item bias and for investigating the causes of item bias.

Summary and Discussion

This paper discussed some important issues for theories in educational measurement and testing. Recent findings in modern learning theory have raised tough challenges to psychometricians. Glaser categorizes these challenges into four main objectives which new achievement measurement should take into account. These

objectives are descriptive, dynamic, structure-oriented, and orchestrate several component tasks.

The pros and cons of two representative approaches of probabilistic modeling commonly used in psychometrics were discussed. Their common, basic principle is that an individual's proficiency level is explained, to a substantial degree, by defining certain human characteristics called traits. These traits are invisible. The only observable outcomes are the students' responses to the test items.

Modern physics and advances in theory and practice of electricity and electronics share this problem with us. We have to infer the outcomes of invisible traits (if they exist), unobservable cognitive processes, individuals' knowledge structures and theory changes from observable responses to test items. Modern physicists have discovered neutrons, electrons and other elementary particles by modeling observable physical phenomena as logical relationships.

It would seem that a new addition to the current theory and techniques of psychometrics is needed to incorporate the new challenges raised by modern learning theory. This addition must overcome an ideological barrier as well as a technological one as stated by Linn (1985). At the same time, the new addition requires an expansion of our common sense to more abstract level, and a re-examination of the basic principles of test theories. However, the time is ripe, and finally research that manifests a new trend has begun to appear in several journals. This includes, for example, renewed concepts of construct validity proposed by Messick, Angoff

and Cronback, and a new technology, called rule space, introduced by Tatsuoka and her associates.

This study introduced a part of the rule space model. The model consists of four major components: (1) Item construction and preparation of the bug library. The bug library consists of response patterns resulting from various sources of misconceptions, sources of incomplete knowledge and erroneous rules of operation; (2) Estimation of parameters, including item parameters of IRT models and bug distributions; (3) Execution of decision rules; (4) Evaluation of the information in the bug library and update of the contents.

Each component requires a substantial amount of time to discuss in detail. The first component was discussed mainly by introducing a method for constructing an item-tree. Making a tree is an application of deterministic relational databases. Each item tree reflects a unique process structure underlying the problem-solving activities. The values of conditional probabilities computed on a specific directed path will help to identify the attribute that causes difficulties in doing the test.

After locating a source of error types, or combinations of several attributes that require special attention in teaching, remediation, or designing instructions, the item-tree program converts them into a list of response patterns. Since the attributes can be production rules, the item tree can be a descriptive representation of a production system.

One of the main differences between the traditional modeling approach and the rule space approach lies in the ways they utilize

the information obtained from a detailed task analysis. The former approach defines a set of new variables and formulates them as parameters in a probability function, or formulates them into probabilistic relationships among probability functions. The rule space approach utilizes algebraic relationships among item response functions for expressing the information obtained from the task analysis. Rules are associated with bug distributions and represented as true points in the rule forms the rule space which is a Cartesian product of two quantities, θ and ζ. Each rule forms the center of an enveloping ellipse with the density of points getting rarer as we depart farther from the center in any direction. Further, the major and minor axis of these ellipses are asymptotically orthogonal (Tatsuoka, 1985). An observed response, on the other hand, will be classified into one of the ellipses if possible. Statistical pattern recognition techniques are applied to classify the observed point. By examining the probability of errors, the student's performance on the test will be diagnosed with an interpretable prescription.

Since the meaning of θ is taken as denoting the levels of proficiencies and not as latent traits or constructs which govern obtaining certain levels of performances on the tests, the change scores resulting from hypothesis-testing activities of testees are explained smoothly without any philosophical difficulties. The model has treated θ as quantitative variable and attributed an important role to the contents of the bug library.

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Table 1
A List of Totally Ordered Sets of the Items Extracted from Figure 1, Method B

1.	6(8)*,2(3)*,1
2.	6(8) ,2(3) ,13
3.	6(8) ,12 ,5
4.	6(8) ,12 ,10
5.	6(8) ,14(16)*,17,13
6.	6(8) ,14(16) ,17,4(11,20)**,18,10
7.	6(8) ,14(16) ,17,4(11,20) ,19
8.	6(8) ,14(16) , 9,7,19

*Items 6 and 8, 2 and 3, 14 and 16 are equivalent.

**Items 4,11, and 20 are equivalent.

Table 2
Slip Probabilities of the First
20 Items for Rules G_2 and G_{13}

Item	G_2 Slip Probability	X_{R_j}	G_{13} Slip Probability	X_{R_j}
1	.043	0	.175	0
2	.025	0	.164	0
3	.025	0	.128	0
4	.154	0	.308	0
5	.285	0	.435	0
6	.610	1	.240	1
7	.006	0	.037	0
8	.495	1	.331	1
9	.430	0	.448	1
10	.002	0	.020	0
11	.012	0	.076	0
12	.269	0	.592	0
13	.001	0	.005	0
14	.192	0	.444	1
15	.006	0	.048	0
16	.208	0	.490	1
17	.003	0	.029	0
18	.021	0	.099	0
19	.000	0	.001	0
20	.001	0	.014	0

Table 3
Bug Distributions of Rules G_{13} and G_2 (N = 1000)

No. of Slips	Frequencies of G_{13}	Frequencies of G_2
0	1	1
1	5	11
2	18	48
3	49	122
4	98	198
5	150	225
6	181	188
7	176	119
8	141	58
9	93	22
10	51	6
11	23	1
12	9	
13	9	

Table 4
The 39 Centroids Representing 39 Different Error Types in Fraction Subtraction Tests (N = 535, n = 40)

Group	θ	ζ	No. of Items	$I(\theta)^{-1}$	Group	θ	ζ	No. of Items	$I(\theta)^{-1}$
1	-2.69	-.80	1	.85	21	.24	-.89	22	.01
2	-1.22	-.69	4	.08	22	-.22	-1.23	14	.02
3	-.75	-.68	8	.05	23	.62	-1.55	32	.01
4	-.46	.75	10	.03	24	1.04	-.61	38	.03
5	.11	.91	18	.02	25	.75	-.05	34	.01
6	.64	1.74	30	.01	26	-.51	-1.62	10	.04
7	-.17	1.48	13	.02	27	-.87	-.56	6	.05
8	.40	-.16	25	.01	28	-1.99	1.01	2	.29
9	.60	-.43	31	.01	29	-.19	1.53	12	.02
10	.57	-.24	29	.01	30	-.24	2.74	10	.03
11	.99	.72	37	.03	31	-1.18	1.46	4	.07
12	1.19	.86	39	.05	32	-1.45	.58	4	.11
13	-.60	-1.58	10	.04	33	.57	-.66	31	.01
14	-.44	-2.31	12	.03	34	.59	-1.39	30	.01
15	-.18	.67	14	.02	35	-1.66	-1.96	4	.16
16	-.08	-1.81	16	.02	36	-.52	-.94	10	.04
17	.16	-.86	20	.02	37	-.32	-1.26	14	.03
18	-.01	-2.12	18	.02	38	-.41	-2.57	13	.03
19	.09	-2.26	20	.02	39	.17	-2.34	22	.01
20	.29	-1.51	24	.01					

Table 5
Frequencies of Students Who Used Either Method A or B

Range of θ	Method A	Frequencies	Method B	Frequencies
$\theta \leq -3$				
$-3 < \theta \leq -2.5$	1	29		
$-2.5 < \theta \leq -2$				
$-2 < \theta \leq -1.5$			36	12
$-1.5 < \theta \leq -1$	2,31	25		
$-1 < \theta \leq -0.5$	3,27	27	13,37	12
$-0.5 < \theta \leq 0$	4,7,29,30	42	14,15,16,18,22,38	39
$0 < \theta \leq 0.5$	5,8	46	17,19,20,21,39	31
$0.5 < \theta \leq 1$	6,9,10,11	78	23,25,34,35	73
$1 < \theta$	12	50	24	26
Total	16 groups	N=306	18 groups	N=193

Method A (Always convert mixed numbers to improper fractions.)

ATTRIBUTES	ITEMS																			
	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
1. Convert a whole number to fraction	0	0	0	0	0	0	1	0	1	0	0	0	0	0	1	0	0	0	1	0
2. Convert 1st mixed number to fraction	0	0	0	1	1	0	1	0	1	1	1	0	1	1	0	1	1	1	1	1
3. Convert 2nd mixed number to fraction	0	0	0	1	1	0	0	0	0	1	1	0	1	1	0	1	0	1	0	1
4. Simplify before subtracting	1	0	0	1	1	0	0	0	0	1	1	1	0	0	0	0	0	1	1	1
5. Find a common denominator	1	1	1	0	1	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0
6. Column borrow to subtract numerator	1	0	0	0	1	0	0	0	1	0	0	0	1	0	0	0	0	1	1	1
7. Reduce answer to simplest form	0	0	0	0	1	0	0	0	1	1	1	1	0	0	1	1	1	1	1	1

Method B (Separate mixed numbers into whole and fraction parts.)

ATTRIBUTES	ITEMS																			
	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
1. Convert a whole number to fraction or mixed number	0	0	0	0	0	0	1	0	0	0	0	0	0	0	1	0	0	0	1	0
2. Separate whole number from fraction	0	0	0	1	1	0	1	0	1	1	1	0	1	1	1	1	1	1	1	1
3. Simplify before getting final answer	0	0	0	1	1	0	0	0	0	1	1	1	0	0	0	0	0	1	1	1
4. Find the common denominator	1	1	1	0	1	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0
5. Borrow 1 from whole number part, change numerator and whole	0	0	0	1	0	0	0	0	0	1	1	0	1	0	0	0	1	1	1	1
6. Column borrow to subtract 2nd numerator from 1st	1	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	1	0	0
7. Reduce answer to simplest form	0	0	0	0	1	0	0	0	0	1	0	1	0	0	0	0	0	0	0	0

Figure 1. Attribute x Item Matrix of Fraction Subtraction Problems by Using Method A or Method B.

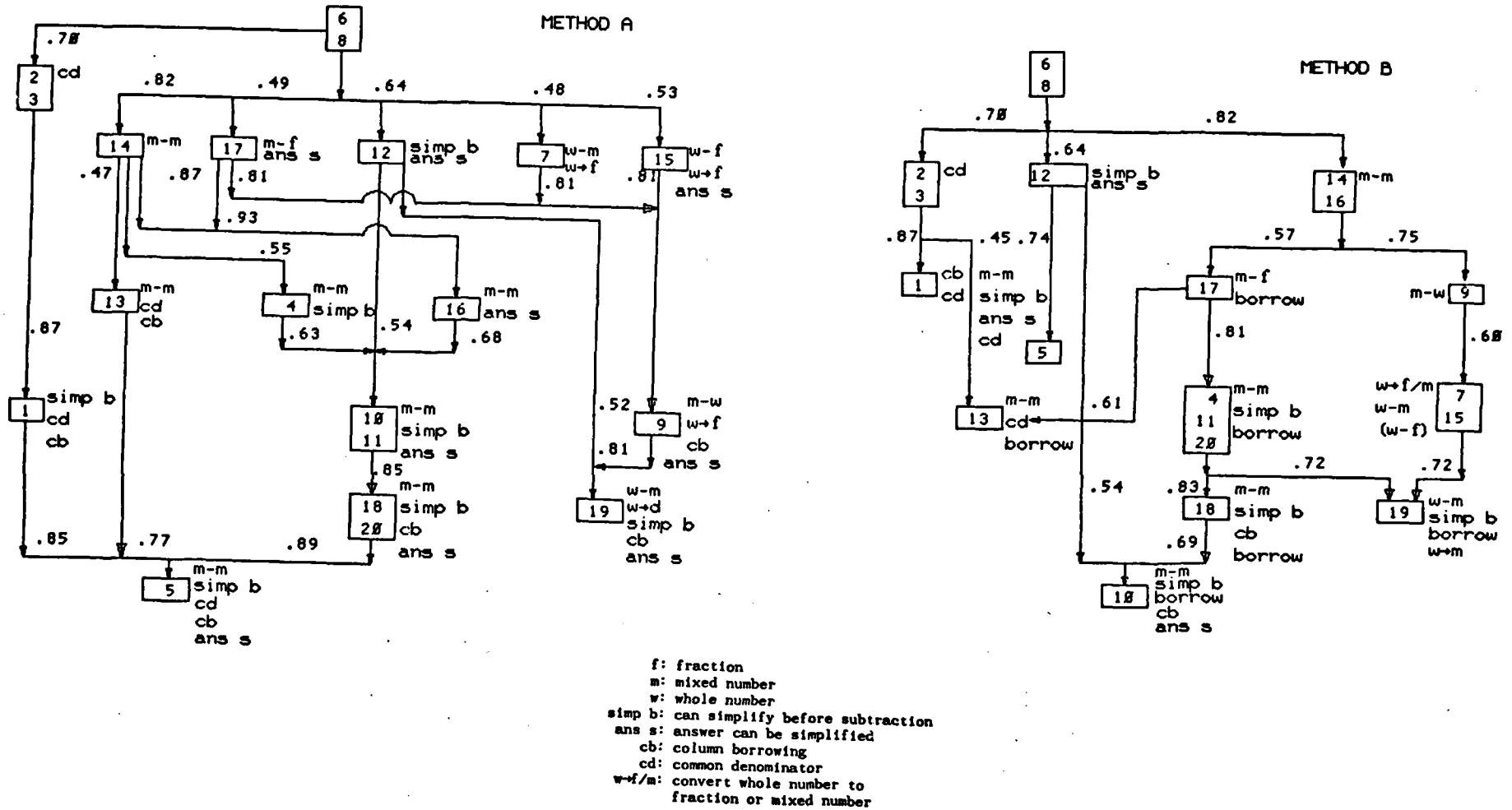


Figure 2. Two Item Trees Constructed from the Attribute x Item Matrices of Method A and B

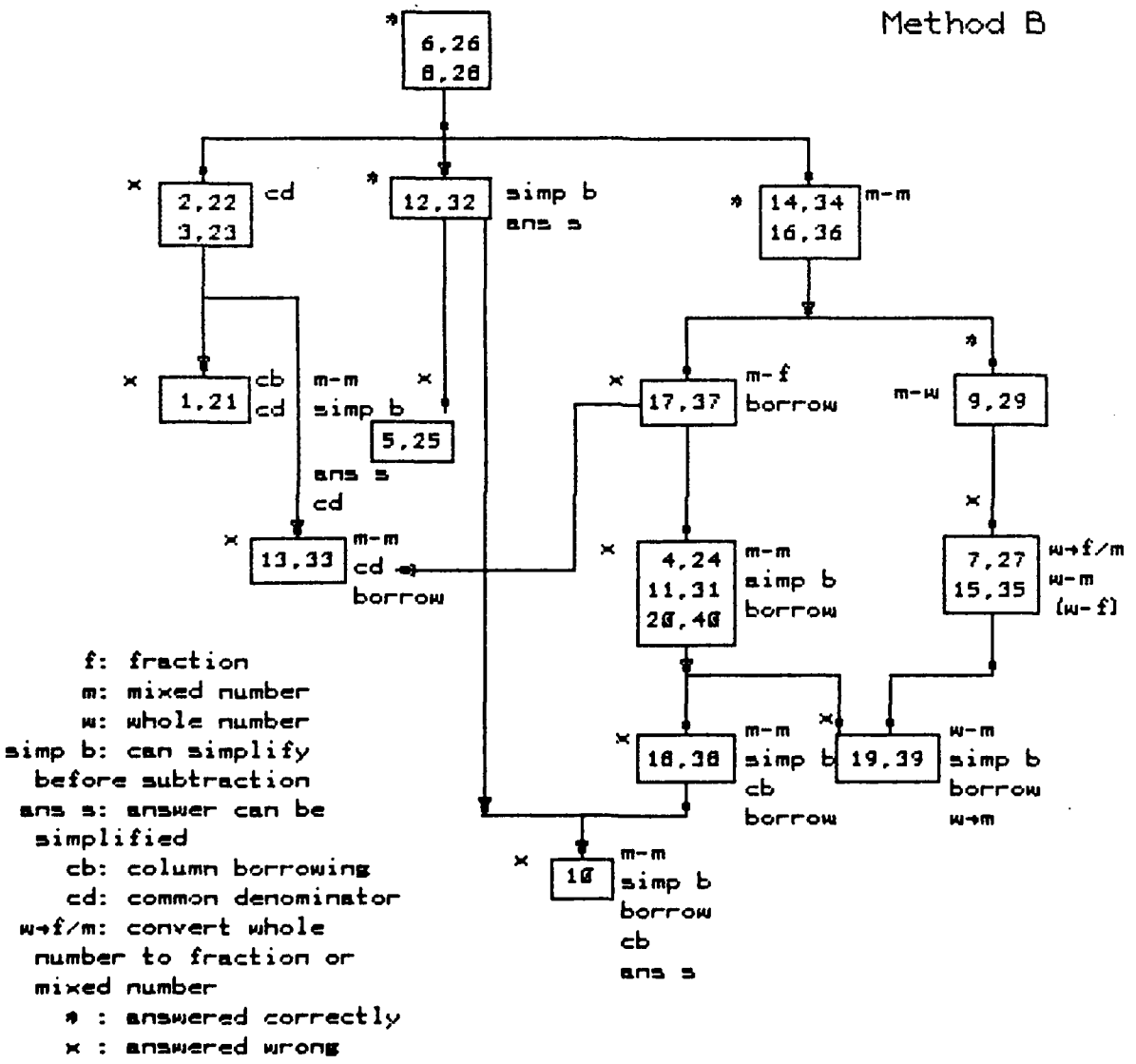


Figure 3. Representation of a rule G14 by the item tree.

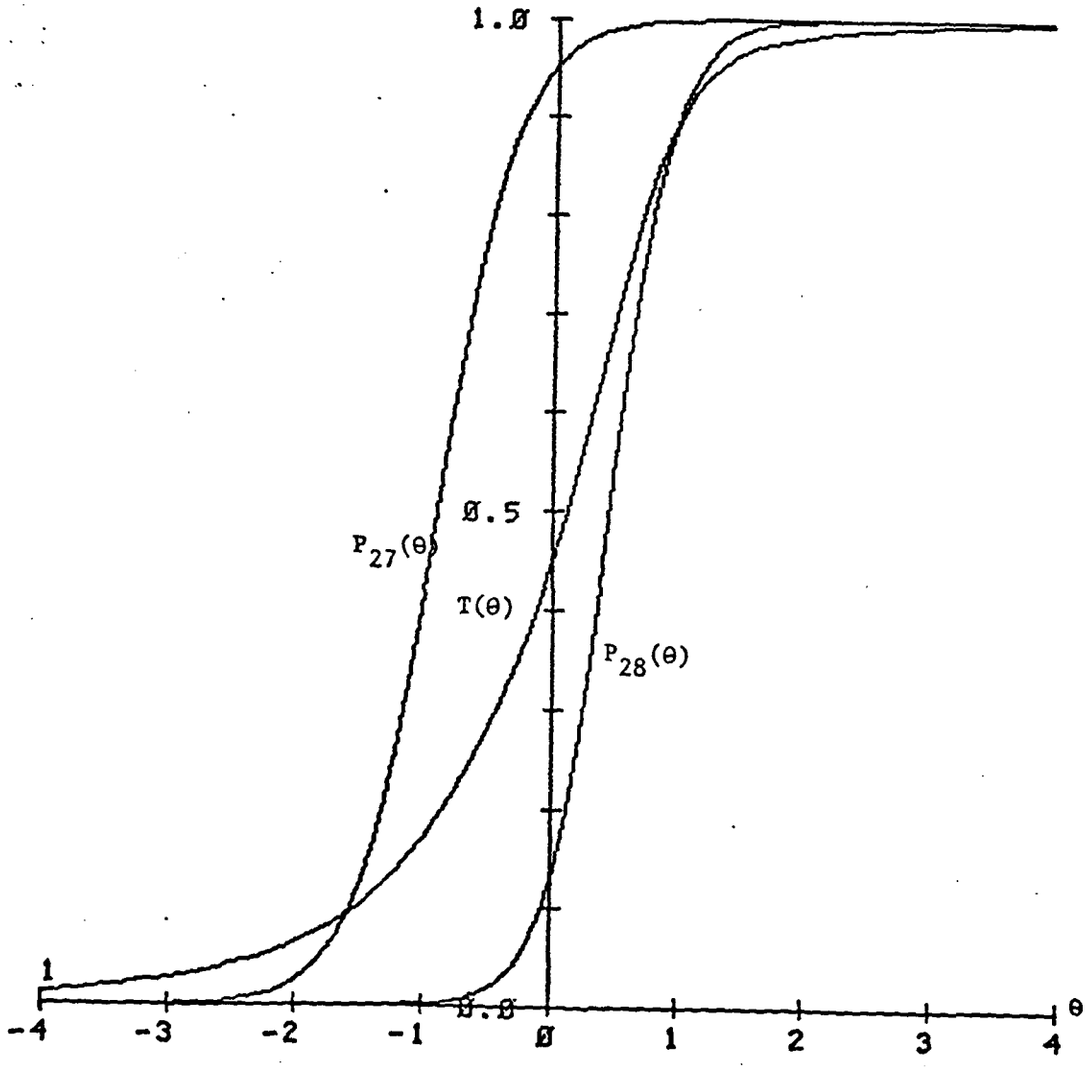


Figure 4. Average of Forty Item Response Curves, T(θ) and IRT Curves of Items 27 and 28.

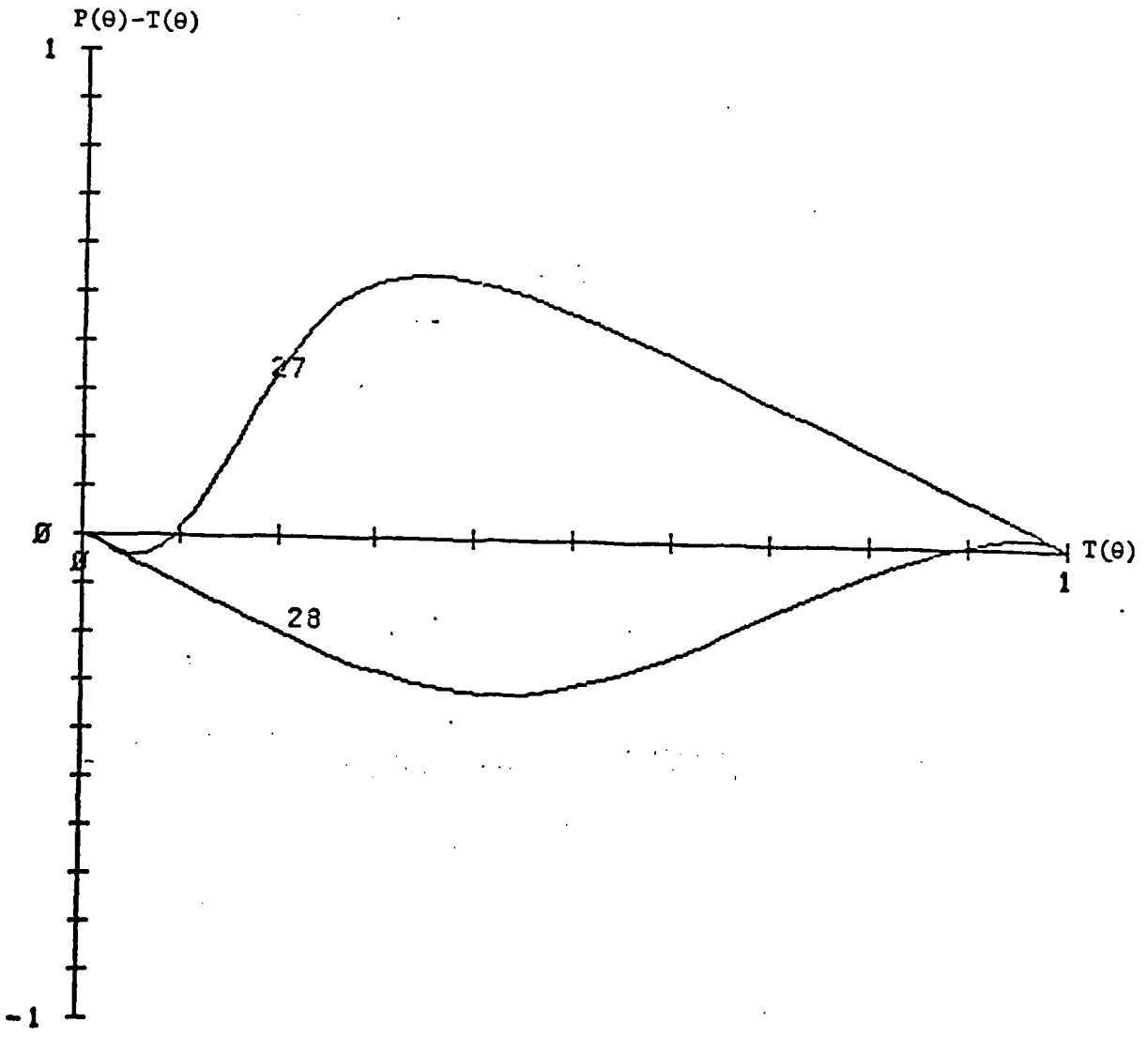


Figure 5. The Curves of $P_{27}(\theta) - T(\theta)$ and $P_{28}(\theta) - T(\theta)$ over $T(\theta)$.

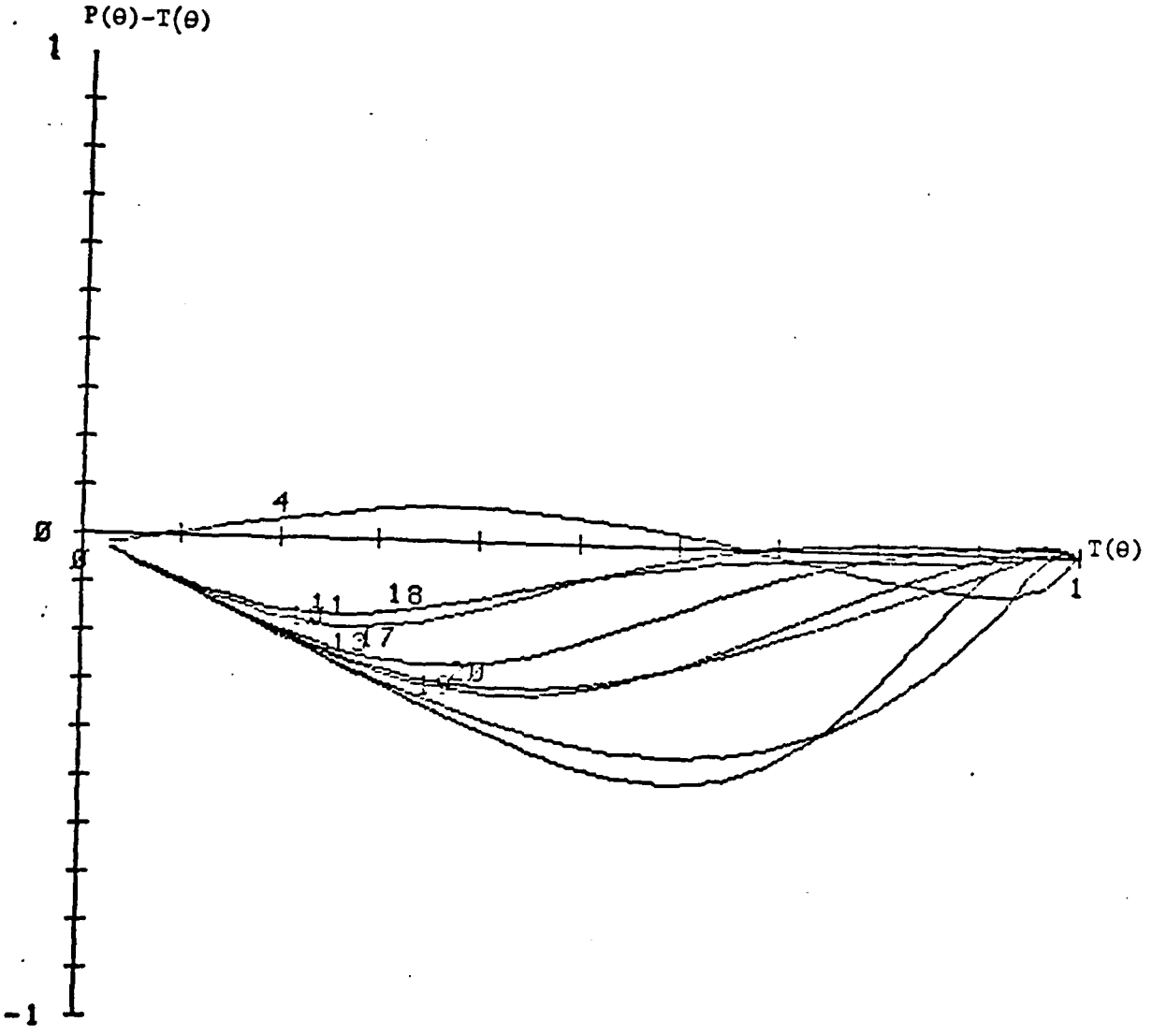


Figure 6. The Curves of Items Requiring Borrowing 4,10,11,13,17, 18,19, and 20.

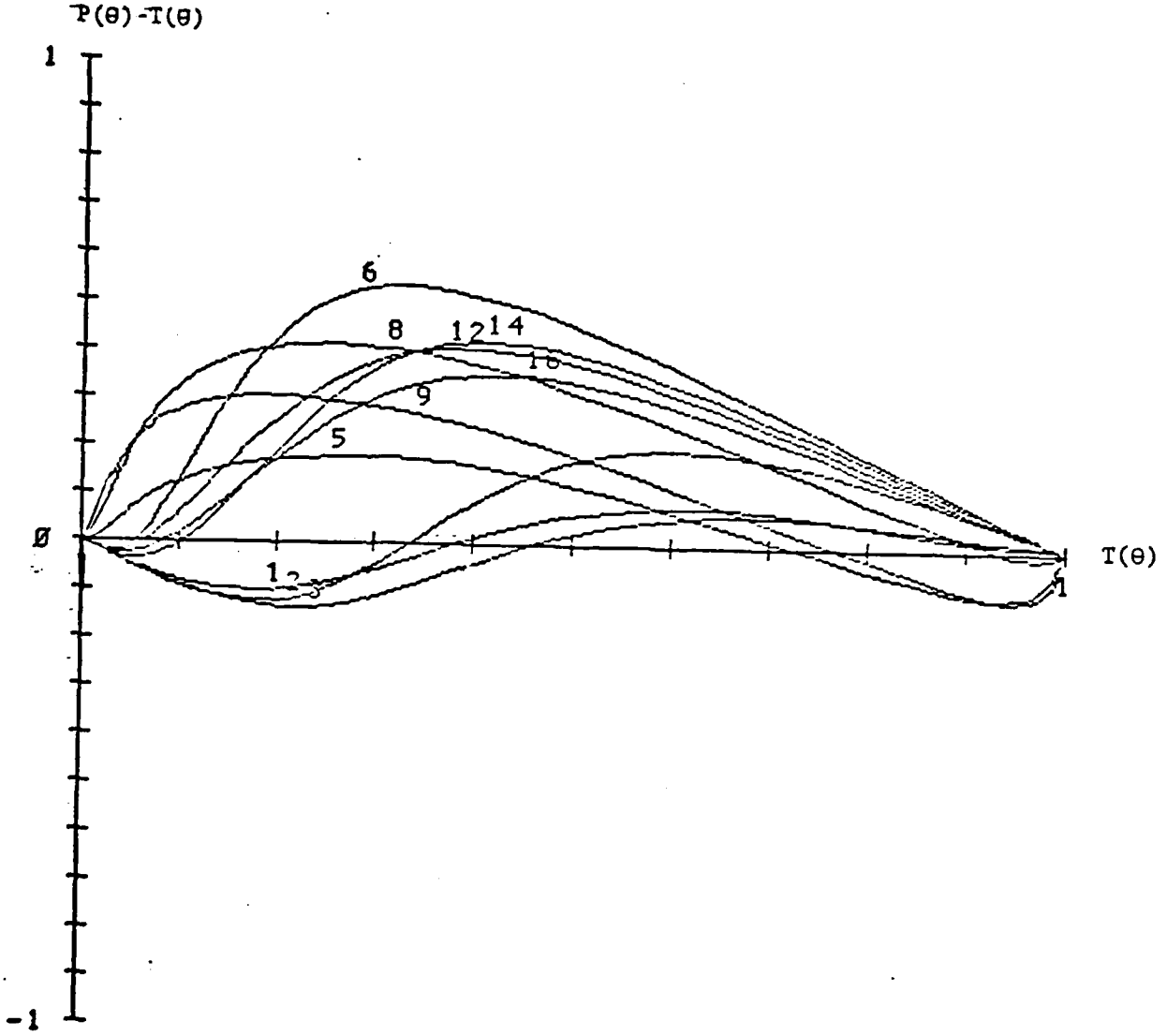


Figure 7. The Curves of Items NOT Requiring Borrowing. They are 1,2,3,5,6,8,9,12,14, and 16.

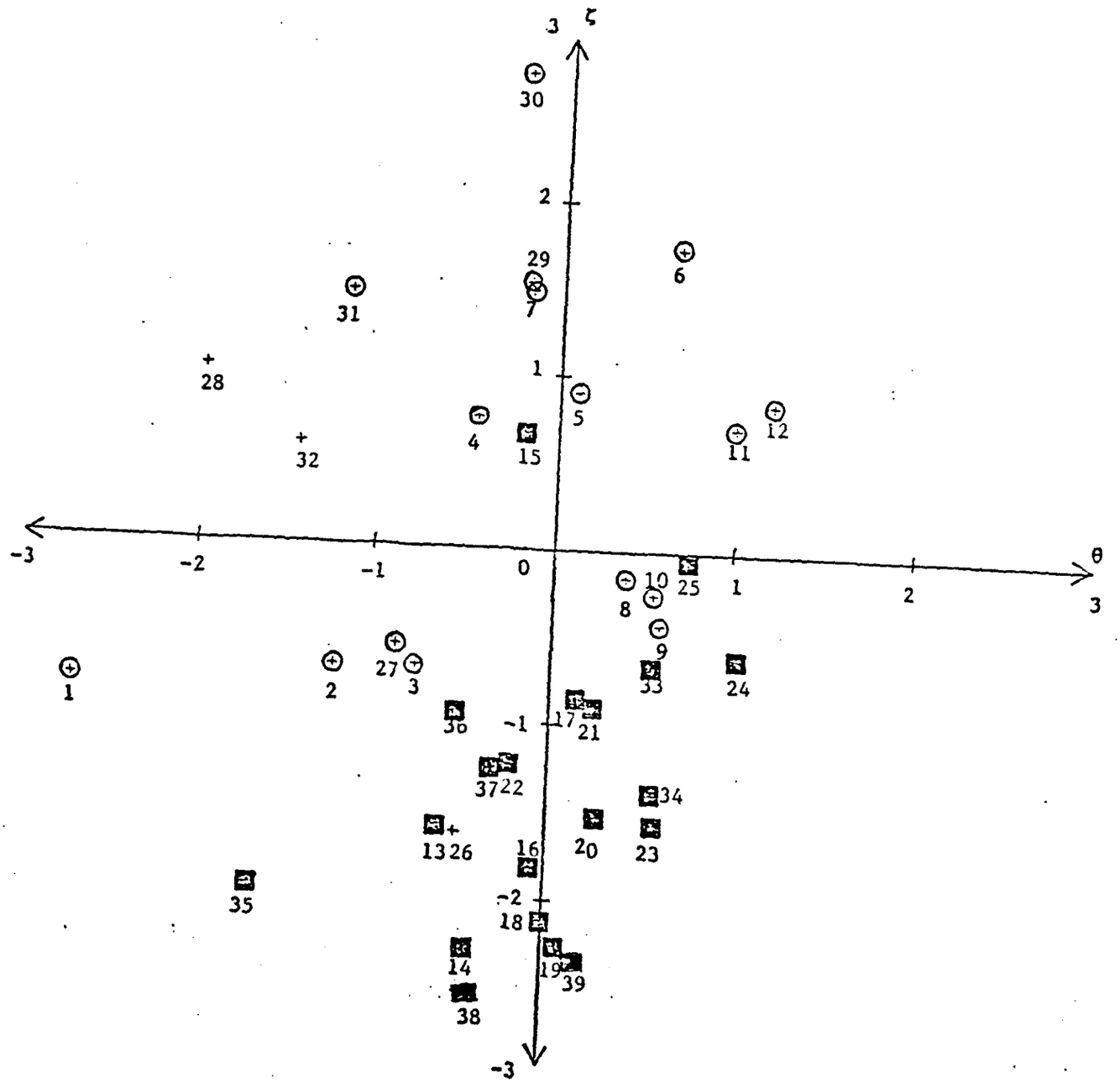


Figure 8. The Thirty-nine Centroids of Thirty-nine Bug Distributions. Black Squares are the Bugs Predicted from Method B. White Circles are from Method A.

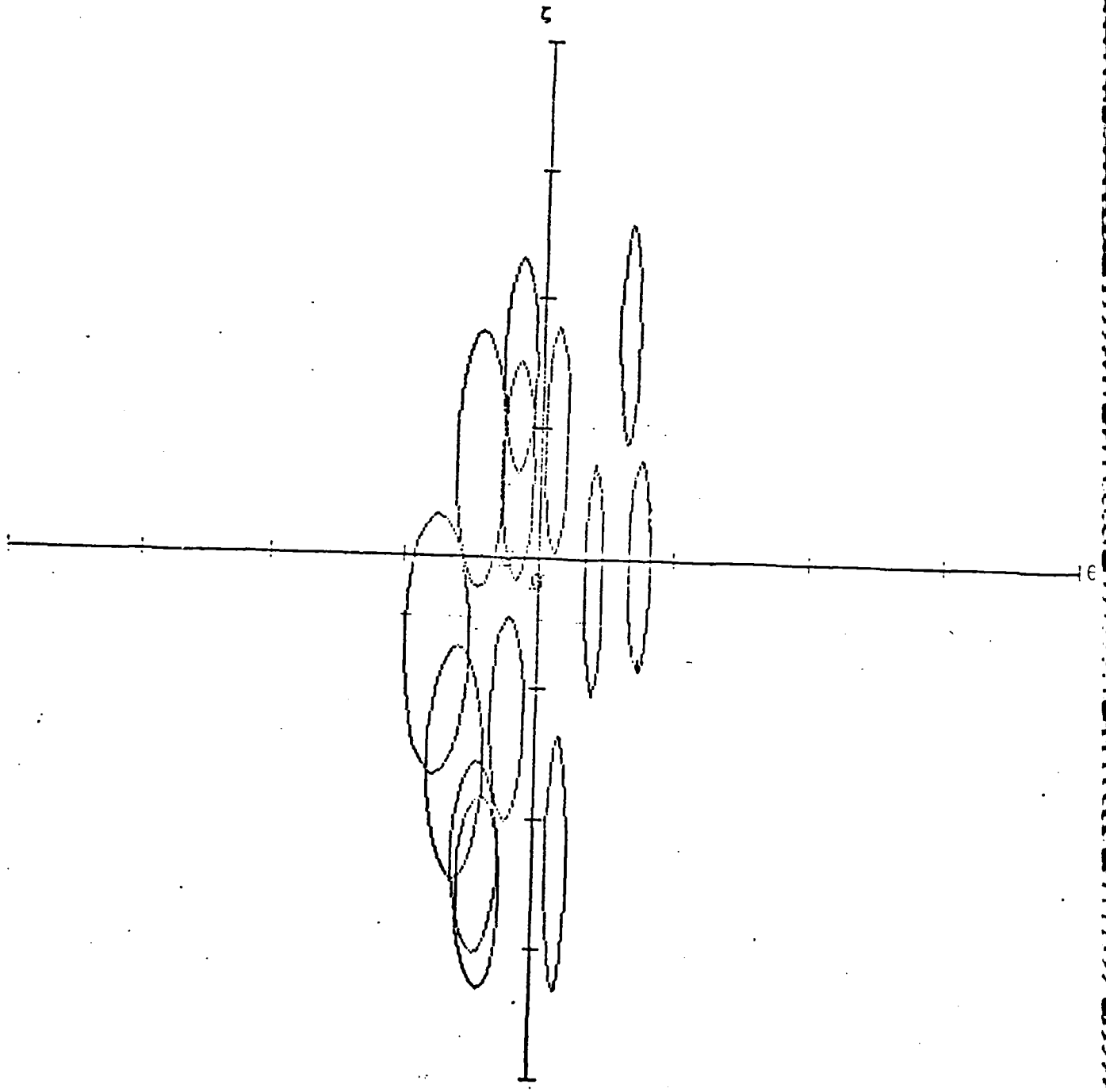
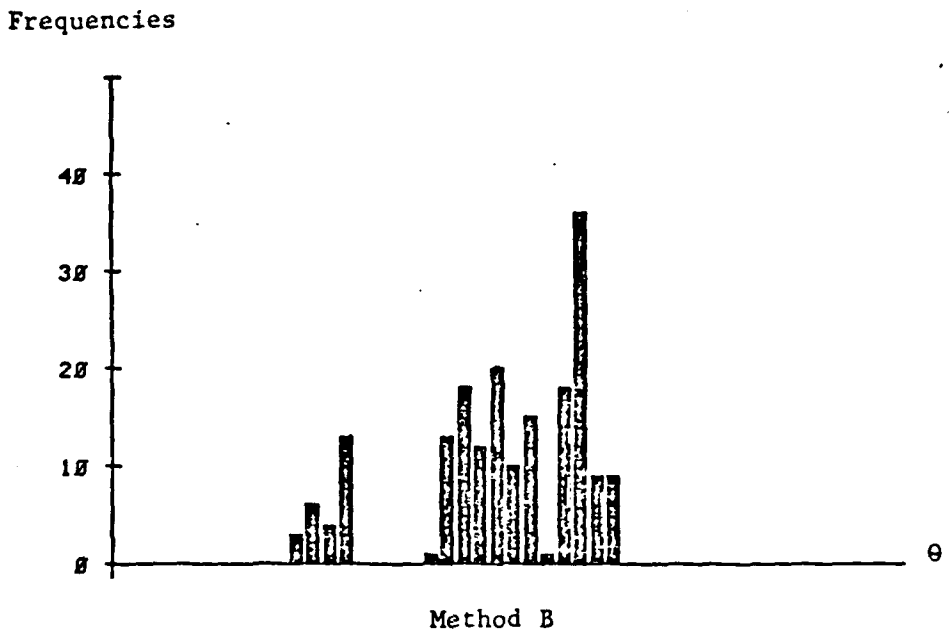


Figure 9. Fifteen Ellipses Representing Fifteen Error Types Randomly Chosen from the Thirty-nine Sets of Ellipses.



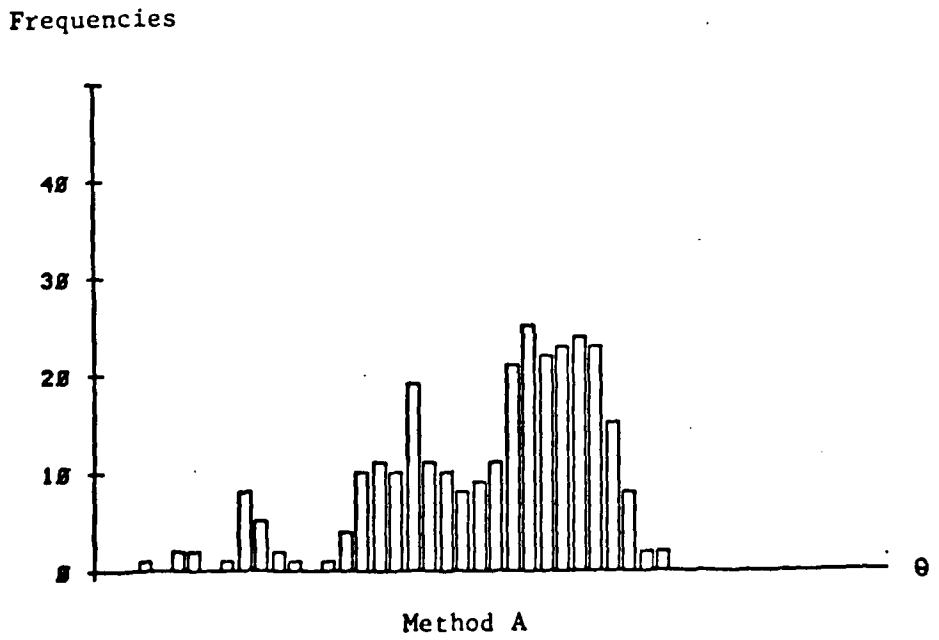


Figure 10. Frequency Distributions of Method A and B Users Over θ.

END

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